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The Gauss map of a hypersurface in Euclidean sphere and the spherical Legendrian duality

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ABSTRACT

We investigate the Gauss map of a hypersurface in Euclidean n-sphere as an application of the theory of Legendrian singularities. We can interpret the image of the Gauss map as the wavefront set of a Legendrian immersion into a certain contact manifold. We interpret the geometric meaning of the singularities of the Gauss map from this point of view.

1. Introduction

In this paper we investigate the Gauss map of a hypersurface in Euclidean n-sphere as an application of the theory of Legendrian singularities. In [3] Izumiya showed a theorem on Legendrian dualities for pseudo-spheres in Minkowski space which has been used as a fundamental tool for the study of the extrinsic differential geometry on submanifolds in these pseudo-spheres from the view point of Singularity theory. We adopt the similar method to [3] for investigating the Gauss map of a hypersurface in Euclidean n-sphere. The pair of the hypersurface and the Gauss map is a Legendrian immersion from the source of the hypersurface to the contact manifold (Δ, K) in the product of the n-spheres which gives the well-known spherical Legendrian duality. This means that the image of Gauss map $\mathbb{G}(U)$ can be interpreted as the wavefront set of the Legendrian immersion

$$\mathcal{L} = (\boldsymbol{x}, \mathbb{G}) : U \to \Delta = \left\{(v, w) \in S^n \times S^n \;\middle|\; v \cdot w = 0\right\}$$

to the contact manifold Δ.

In Section 2 we introduce the basic notion of the spherical Gauss map, the shape operator and the Gauss–Kronecker curvature of a hypersurface in Euclidean n-sphere. We show in Section 3 that (Δ, K) is a contact manifold and $\mathcal{L}$ is a Legendrian immersion, and the image of Gauss map $\mathbb{G}(U)$ is interpreted as the wavefront set of the Legendrian immersion $\mathcal{L}$. In Sections 4 and 5, we introduce the family of height functions on a hypersurface in Euclidean n-sphere and show that the height function is a generating family of the Legendrian immersion $\mathcal{L}$. In Section 6, we apply the theory of Legendrian singularities and consider the geometric meaning of the singularities of the Gauss map by using the theory of contact between

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hypersurfaces and hypersphere due to Montaldi [5]. In Section 7 we study generic properties for $n \leqslant 6$. We briefly review the results on surfaces in S^3 of Miyawaki [6] in Section 8.

2. Basic notations

In this section we prepare basic notions on hypersurface in Euclidean sphere.

Let $S^n = \{\boldsymbol{V} \in \mathbb{R}^{n+1} \mid \boldsymbol{V} \cdot \boldsymbol{V} = 1\}$ be an n-dimensional Euclidean sphere, where $\boldsymbol{V} \cdot \boldsymbol{V}$ is the canonical inner product. Given a vector $\boldsymbol{v} \in \mathbb{R}^{n+1} \setminus \{0\}$ and a real number c, the hyperplane with normal $\boldsymbol{v}$ is given by

$$HP(\boldsymbol{v}, c) = \{\boldsymbol{w} \in \mathbb{R}^{n+1} \mid \boldsymbol{v} \cdot \boldsymbol{w} = c\}.$$

The hypersphere in S^n is given by

$$S^{n-1}(\boldsymbol{v}, c) = S^n \cap H(\boldsymbol{v}, c) = \{\boldsymbol{w} \in S^n \mid \boldsymbol{v} \cdot \boldsymbol{w} = c\}.$$

We say that $S^{n-1}(\boldsymbol{v}, c)$ is a *great hypersphere* if $c = 0$, a *small hypersphere* if $c \neq 0$. For any $\boldsymbol{a}_{(i)} = (a^0_{(i)}, \dots, a^n_{(i)}) \in \mathbb{R}^{n+1}$ $(i = 1, 2, \dots, n)$, the wedge product $\boldsymbol{a}_{(1)} \wedge \boldsymbol{a}_{(2)} \wedge \cdots \wedge \boldsymbol{a}_{(n)}$ is defined by

$$\boldsymbol{a}_{(1)} \wedge \boldsymbol{a}_{(2)} \wedge \cdots \wedge \boldsymbol{a}_{(n)} = \det \begin{pmatrix} \boldsymbol{e}_{(0)} & \boldsymbol{e}_{(1)} & \cdots & \boldsymbol{e}_{(n)} \\ a^0_{(1)} & a^1_{(1)} & \cdots & a^n_{(1)} \\ a^0_{(2)} & a^1_{(2)} & \cdots & a^n_{(2)} \\ \vdots & \vdots & \cdots & \vdots \\ a^0_{(n)} & a^1_{(n)} & \cdots & a^n_{(n)} \end{pmatrix},$$

where $\{\boldsymbol{e}_{(0)}, \dots, \boldsymbol{e}_{(n+1)}\}$ is the canonical basis of $\mathbb{R}^{n+1}$. We can easily show that

$$\boldsymbol{b} \cdot (\boldsymbol{a}_{(1)} \wedge \boldsymbol{a}_{(2)} \wedge \cdots \wedge \boldsymbol{a}_{(n)}) = \det(\boldsymbol{b}, \boldsymbol{a}_{(1)}, \dots, \boldsymbol{a}_{(n)}),$$

so that $\boldsymbol{a}_{(1)} \wedge \boldsymbol{a}_{(2)} \wedge \cdots \wedge \boldsymbol{a}_{(n)}$ is orthogonal to any $\boldsymbol{a}_{(i)}$ $(i = 1, 2, \dots, n)$.

Let $\boldsymbol{x} : U \to S^n$ be a regular hypersurface (i.e. an embedding), where $U \subset \mathbb{R}^{n-1}$ is an open subset. We denote that $M = \boldsymbol{x}(U)$ and identify M and U through the embedding $\boldsymbol{x}$. Since $\boldsymbol{x}(u) \cdot \boldsymbol{x}(u) = 1$ for any $u = (u_1, u_2, \dots, u_{n-1}) \in U$, we have $\boldsymbol{x}_{u_i}(u) \cdot \boldsymbol{x}(u) = 0$ where $\boldsymbol{x}_{u_i}(u) = \frac{\partial \boldsymbol{x}}{\partial u_i}(u)$.

For any point $u \in U$, we define the unit normal vector $\boldsymbol{e}(u)$ of M at $\boldsymbol{x}(u)$ by

$$\boldsymbol{e}(u) = \frac{\boldsymbol{x}(u) \wedge \boldsymbol{x}_{u_1}(u) \wedge \boldsymbol{x}_{u_2}(u) \wedge \cdots \wedge \boldsymbol{x}_{u_n}(u)}{\|\boldsymbol{x}(u) \wedge \boldsymbol{x}_{u_1}(u) \wedge \boldsymbol{x}_{u_2}(u) \wedge \cdots \wedge \boldsymbol{x}_{u_n}(u)\|}.$$

Then we define a map $\mathbb{G} : U \to S^n$ by $\mathbb{G}(u) = \boldsymbol{e}(u)$ which is called the *Gauss map* on the hypersurface $M = \boldsymbol{x}(U)$.

By definition, we have

$$\boldsymbol{e}(u) \cdot \boldsymbol{e}(u) = 1, \qquad \boldsymbol{x}(u) \cdot \boldsymbol{e}(u) = \boldsymbol{x}_{u_i}(u) \cdot \boldsymbol{e}(u) = 0 \quad (\forall u \in U).$$

Since $\boldsymbol{x}$ is an embedding, $\{\boldsymbol{x}_{u_1}(u), \dots, \boldsymbol{x}_{u_{n-1}}(u)\}$ is linearly independent for any $u \in U$. Therefore, we obtain a basis

$$\{\boldsymbol{x}(u), \boldsymbol{x}_{u_1}(u), \dots, \boldsymbol{x}_{u_{n-1}}(u), \boldsymbol{e}(u)\}$$

of $T_p\mathbb{R}^{n+1}$, where $p = \boldsymbol{x}(u)$.

We investigate the extrinsic differential geometry of hypersurfaces in n-sphere by using the Gauss map $\mathbb{G}$ of $M = \boldsymbol{x}(U)$ which plays similar roles to those of the Gauss map of a hypersurface in Euclidean space. Taking the derivatives of the equalities $\boldsymbol{e} \cdot \boldsymbol{e} = 1$ and $\boldsymbol{x} \cdot \boldsymbol{e} = 0$, we have

$$d(\boldsymbol{e} \cdot \boldsymbol{e})(u) = 2d\boldsymbol{e}(u) \cdot \boldsymbol{e}(u) = 0,$$
$$d(\boldsymbol{x} \cdot \boldsymbol{e})(u) = d\boldsymbol{x}(u) \cdot \boldsymbol{e}(u) + \boldsymbol{x}(u) \cdot d\boldsymbol{e}(u) = \boldsymbol{x}(u) \cdot d\boldsymbol{e}(u) = 0$$

for any $u \in U$. Then we have the following lemma:

Lemma 2.1. *For any $p = \boldsymbol{x}(u_0) \in M$, the derivative $d\mathbb{G}(u_0)$ is a linear transformation on the tangent space T_pM.*

We define the notion of curvatures as follows: We call the linear transformation $S_p = -d\mathbb{G}(u_0) : T_pM \to T_pM$ the *shape operator* of $M = \boldsymbol{x}(U)$ at $p = \boldsymbol{x}(u_0)$. We denote the eigenvalues of S_p by $\kappa_i(p)$ $(i = 1, \dots, n-1)$, which are called the *principal curvatures* of $M = \boldsymbol{x}(U)$ at $p = \boldsymbol{x}(u_0)$.

The (*spherical*) *Gauss–Kronecker curvature* of $M = \boldsymbol{x}(U)$ at $p = \boldsymbol{x}(u_0)$ is defined to be

$$K(u_0) = \det S_p.$$

We say that a point $u \in U$ or $p = \boldsymbol{x}(u)$ is an *umbilic point* if there exists $\kappa(p) \in \mathbb{R}$ such that $S_p = \kappa(p) \cdot 1_{T_pM}$. If all points on M are umbilic, we say that $M = \boldsymbol{x}(U)$ is *totally umbilic*. The following properties are well-known results, so that we omit the proof.

Lemma 2.2. *Suppose that $M = \boldsymbol{x}(U)$ is totally umbilic. Then $\kappa(p)$ is constant κ. Under this condition, we have the following classification*:

(1) *If $\kappa = 0$, then M is a part of a great hypersphere.*
(2) *If $\kappa \neq 0$, then M is a part of a small hypersphere.*

In the last part of this section, we prove the spherical Weingarten formula. We define the *first fundamental invariant* by $g_{ij}(u) = \boldsymbol{x}_{u_i} \cdot \boldsymbol{x}_{u_j}$ and the *second fundamental invariant* by $h_{ij} = \boldsymbol{x}_{u_i u_j} \cdot \boldsymbol{e} = -\boldsymbol{x}_{u_i} \cdot \boldsymbol{e}_{u_j}$ for any $u \in U$.

Proposition 2.3. *Under the above notation, we have the following spherical Weingarten formula*:

$$-\mathbb{G}_{u_i} = \sum_{j=1}^{n-1} h_i^j \boldsymbol{x}_{u_j} \quad (i = 1, \dots, n-1),$$

where $(h_i^j)_{i,j} := (h_{ij})_{i,j}(g^{ij})_{i,j}$ and $(g^{ij}) = (g_{ij})^{-1}$.

Proof. Since $\boldsymbol{e}_{u_i} \cdot \boldsymbol{e} = \boldsymbol{e}_{u_i} \cdot \boldsymbol{x} = 0$, there exist real numbers Γ_i^j such that

$$\boldsymbol{e}_{u_i} = \sum_{j=1}^{n-1} \Gamma_i^j \boldsymbol{x}_{u_j}(u).$$

By definition, we have

$$-h_{ik}(u) = \boldsymbol{e}_i(u) \cdot \boldsymbol{x}_k(u) = \sum_{j=1}^{n-1} \Gamma_i^j \boldsymbol{x}_j(u) \cdot \boldsymbol{x}_k(u) = \sum_{j=1}^{n-1} \Gamma_i^j g_{jk}(u).$$

It follows that

$$-h_i^j(u) = \sum_{k=1}^{n-1} (-h_{ik}) g^{kj} = \sum_{k=1}^{n-1} \left(\sum_{l=1}^{n-1} \Gamma_i^l g_{jl}(u) \right) g^{kj} = \sum_{l=1}^{n-1} \Gamma_i^l \delta_{jl} = \Gamma_i^j.$$

Therefore we have $\Gamma_i^j = -h_i^j(u)$. This completes the proof. □

As a corollary of the above proposition, we have an explicit expression for the spherical Gauss–Kronecker curvature by the first and second fundamental invariants.

Corollary 2.4. *Under the same notation as in the above proposition, the spherical Gauss–Kronecker curvature is given by*

$$K_p = \frac{\det(h_{ij}(u))}{\det(g_{ij}(u))}.$$

Proof. By the spherical Weingarten formula, the representation matrix of the shape operator S_p with respect to the basis $\{\boldsymbol{x}_{u_1}, \dots, \boldsymbol{x}_{u_{n-1}}\}$ is $(h_i^j)_{i,j} = (h_{ij})_{i,j}(g^{ij})_{i,j}$. It is follows from this fact that

$$K_p = \det\bigl(h_i^j(u)\bigr) = \det\bigl(h_{ij}(u)\bigr) \times \det\bigl(g_{ij}(u)\bigr)^{-1} = \frac{\det(h_{ij}(u))}{\det(g_{ij}(u))}. \qquad \square$$

We say that a point $p = \boldsymbol{x}(u)$ is a *parabolic point* of $M = \boldsymbol{x}(U)$ if $K(u) = 0$.

3. The spherical Gauss map as a wave front

In this section we show that the pair $(\boldsymbol{x}, \mathbb{G})$ is a Legendrian immersion from U to a certain contact manifold and the Gauss map $\mathbb{G}(U)$ can be considered as a wave front. We now briefly review some properties of contact manifolds and Legendrian submanifolds. Let W be a $2n+1$-dimensional smooth manifold and K be a tangent hyperplane field on W. Locally such a field is defined as the field of zeros of a 1-form α. If tangent hyperplane field K is non-degenerate, we say that (W, K) is a *contact manifold*. Here K is said to be *non-degenerate* if $\alpha \wedge (d\alpha)^n \neq 0$ at any point of W. In this case K is called a *contact structure* and α is a *contact form*. Let $\phi : W \to W'$ be a diffeomorphism between contact manifolds (W, K) and (W', K'). We say that ϕ is a *contact diffeomorphism* if $d\phi(K) = K'$. Two contact manifolds (W, K) and (W', K') are *contact diffeomorphic* if there exists a contact diffeomorphism $\phi : W \to W'$. A submanifold $i : L \subset W$ of a contact manifold (W, K) is a *Legendrian submanifold* if $\dim L = n$ and $di_p(T_pL) \subset K_{i(p)}$ at any point $p \in L$. We consider a smooth fiber bundle $\pi : N \to A$. The fiber bundle $\pi : N \to A$ is called a *Legendrian fibration* if its total space W is furnished with a contact structure and its fibers are Legendrian submanifolds. Let $\pi : N \to A$ be a Legendrian fibration. For a Legendrian submanifold $i : L \subset N$, a map $\pi \circ i : L \to A$ is called a *Legendrian map*. The image of the Legendrian map $\pi \circ i$ is called a *wavefront set* of i which is denoted by $W(i)$. For any $p \in W$, it is known that there is a local coordinate system $(x_i, \ldots, x_n, p_1, \ldots, p_n, z)$ around p such that $\pi(x_i, \ldots, x_n, p_1, \ldots, p_n, z) = (x_i, \ldots, x_n, z)$ and the contact structure is given by the 1-form $\alpha = dz - \sum_{i=1}^n p_i dx_i$ (cf., [1], Part III).

We now consider the following double fibrations of S^n:

$$\Delta = \{(\boldsymbol{v}, \boldsymbol{w}) \in S^n \times S^n \mid \boldsymbol{v} \cdot \boldsymbol{w} = 0\},$$
$$\pi_1 : \Delta \ni (\boldsymbol{v}, \boldsymbol{w}) \mapsto \boldsymbol{v} \in S^n, \qquad \pi_2 : \Delta \ni (\boldsymbol{v}, \boldsymbol{w}) \mapsto \boldsymbol{w} \in S^n,$$
$$\theta_1 = d\boldsymbol{v} \cdot \boldsymbol{w}|_\Delta, \qquad \theta_2 = \boldsymbol{v} \cdot d\boldsymbol{w}|_\Delta.$$

Here, $d\boldsymbol{v} \cdot \boldsymbol{w} = \sum_{i=0}^n w_i dv_i$ and $\boldsymbol{v} \cdot d\boldsymbol{w} = \sum_{i=0}^n v_i dw_i$. Since $d(\boldsymbol{v} \cdot \boldsymbol{w}) = d\boldsymbol{v} \cdot \boldsymbol{w} + \boldsymbol{v} \cdot d\boldsymbol{w}$ and $\boldsymbol{v} \cdot \boldsymbol{w} = 0$ on Δ, $\theta_1^{-1}(0)$ and $\theta_2^{-1}(0)$ define the same tangent hyperplane field over Δ which is denoted by K.

Theorem 3.1. *Under the above notation, (Δ, K) is a contact manifold and both of π_i are Legendrian fibrations.*

By definition, Δ is a smooth submanifold in $\mathbb{R}^{n+1} \times \mathbb{R}^{n+1}$ and each π_i $(i = 1, 2)$ is a smooth fibration. It is well known that (Δ, K) is a contact manifold. Therefore, we omit the detailed proof. Moreover, by the definition of the contact form θ_1, θ_2, all fibers of π_1 and π_2 are Legendrian submanifolds in (Δ, K).

We now interpret the Gauss map of a hypersurface in S^n as a wave front set in the above contact manifold. For any regular hypersurface $\boldsymbol{x} : U \to S^n$, we have $\boldsymbol{x} \cdot \boldsymbol{e} = 0$. Therefore we can define an embedding $\mathcal{L} : U \to \Delta$ by $\mathcal{L}(u) = (\boldsymbol{x}(u), \boldsymbol{e}(u)) = (\boldsymbol{x}(u), \mathbb{G}(u))$.

Proposition 3.2. *The mapping $\mathcal{L}$ is a Legendrian embedding to the contact manifold (Δ, K).*

Proof. Since $\boldsymbol{x} : U \to S^n$ is an embedding, $\mathcal{L}$ is also an embedding and $dim(\mathcal{L}(U)) = n - 1$. Since $\mathcal{L}^*\theta_1 = d\boldsymbol{x} \cdot \boldsymbol{e} = 0$, $\mathcal{L}$ is a Legendrian embedding. This completes the proof. □

By definition, we have $\pi_2 \circ \mathcal{L}(U) = \mathbb{G}(U)$. Then we have the following corollary:

Corollary 3.3. *For any hypersurface $\boldsymbol{x} : U \to S_n$, $\mathbb{G}(U)$ is a wave front set of $\mathcal{L}$ with respect to the Legendrian fibration π_2.*

4. Spherical height functions

In this section we introduce a family of functions on a hypersurface in the sphere which is useful for the study of singularities of the spherical Gauss map. Let $\boldsymbol{x} : U \to S^n$ be a hypersurface. We define a family of functions

$$H : U \times S^n \to \mathbb{R}$$

by $H(u, V) = \boldsymbol{x}(u) \cdot V$. We call H is a *height function* on $\boldsymbol{x} : U \to S^n$.

Proposition 4.1. *Let $H : U \times S^n \to \mathbb{R}$ be a height function on $\boldsymbol{x} : U \to S^n$. Then $H(u, V) = \partial H(u, V)/\partial u_i = 0$ (for $i = 1, 2, \ldots, n-1$) if and only if $V = \pm\mathbb{G}(u)$.*

Proof. Since $\{\boldsymbol{x}(u), \boldsymbol{x}_{u_1}(u), \ldots, \boldsymbol{x}_{u_{n-1}}(u), \boldsymbol{e}(u)\}$ is a basis of the vector space $T_p\mathbb{R}^{n+1}$ where $p = \boldsymbol{x}(u)$, there exist real numbers $\alpha, \beta_1, \ldots, \beta_{n-1}, \gamma$ such that $V = \alpha\boldsymbol{x}(u) + \sum_{j=1}^{n-1} \beta_j \boldsymbol{x}_{u_j}(u) + \gamma\boldsymbol{e}(u)$ where $\alpha^2 + \sum_{j=1}^{n-1} \beta_j^2 + \gamma^2 = 1$. Therefore $H(u, V) = \boldsymbol{x}(u) \cdot V = 0$ if and only if $\boldsymbol{x}(u) \cdot V = \alpha = 0$. Since $0 = \partial H(u, V)/\partial u_i = \boldsymbol{x}_{u_i}(u) \cdot V$, we have $\sum_{j=1}^{n-1} \beta_j g_{ij} = 0$ (for $i = 1, 2, \ldots, n-1$), where g_{ij} are first fundamental invariants. It follows from the fact that $(g_{ij})_{i,j}$ is positive definite, we have $\beta_i = 0$ (for $i = 1, 2, \dots, n-1$). This completes the proof. □

The above proposition means that the image of Gauss map of hypersurface $\boldsymbol{x} : U \to S^n$ coincides with the discriminant set D_H of the height function on $M = \boldsymbol{x}(U)$ (cf., Appendix of Izumiya [3]). For a given $V_0 \in S^n$, we define the *height function with the direction* V_0 by $h_{V_0}(u) = H(u, V_0)$ and we denote the *Hessian matrix* of the height function with the direction V_0 at u_0 by $\operatorname{Hess}(h_{V_0})(u)$.

Lemma 4.2. *Let $H : U \times S^n \to \mathbb{R}$ be a height function on $\boldsymbol{x} : U \to S^n$ and $V_0 = \pm\mathbb{G}(u_0)$. Then $p = \boldsymbol{x}(u_0)$ is a parabolic point if and only if* $\det \operatorname{Hess}(h_{V_0})(u_0) = 0$.

Proof. By definition, we have

$$\operatorname{Hess}(h_{V_0})(u_0) = \big(\boldsymbol{x}_{u_i u_j}(u_0) \cdot \pm\boldsymbol{e}(u_0)\big) = \pm\big(h_{ij}(u_0)\big).$$

By Corollary 2.5, we have

$$K_p = \frac{\det \operatorname{Hess}(h_{V_0}(u_0))}{\det(g_{ij}(u_0))}.$$

The above assertion follows from this formula. □

Lemma 4.3. *Let $H : U \times S^n \to \mathbb{R}$ be a height function on $\boldsymbol{x} : U \to S^n$ and $V_0 = \pm\mathbb{G}(u_0)$. Then $p = \boldsymbol{x}(u_0)$ is an umbilic point with $\kappa(p) = 0$ if and only if* $\operatorname{rank} \operatorname{Hess}(h_{V_0})(u_0) = 0$, *where $\kappa(p)$ is the principal curvature.*

Proof. By Proposition 4.1, $V_0 \in D_H$ if and only if there exist $u_0 \in U$ such that $V_0 = \pm\mathbb{G}(u_0)$. On the other hand, by the Weingarten formula, $p = \boldsymbol{x}(u_0)$ is an umbilic point if and only if there exists an orthogonal matrix A such that ${}^tA(h_i^j(u_0))A = \kappa(p)I$. Therefore, we have $(h_i^j) = A\kappa(p)I\,{}^tA = \kappa(p)I$, so that $(h_{ij}(u_0)) = \kappa(p)(g_{ij}(u_0))$. It follows that $(h_{ij}(u_0))$ is the zero matrix if and only if $\kappa(p) = 0$. Moreover, we have $\operatorname{Hess}(h_{V_0})(u_0) = \pm(h_{ij}(u_0))$. This completes the proof. □

We say that $p = \boldsymbol{x}(u_0)$ is a *geodesic point* (or, *flat umbilic point*) if it is an umbilic point with $\kappa(p) = 0$.

5. Generating families of Legendrian immersions

In this section we show that the height function $H : U \times S^n \to \mathbb{R}$ is the generating family of $\mathcal{L}$ at least locally.

Let $F : (\mathbb{R}^k \times \mathbb{R}^n, 0) \to (\mathbb{R}, 0)$ be a function germ. We say that F is a *Morse family of hypersurfaces* if the mapping

$$\Delta^* F = \left(F, \frac{\partial F}{\partial q_1}, \dots, \frac{\partial F}{\partial q_k}\right) : \big(\mathbb{R}^k \times \mathbb{R}^n, 0\big) \to \big(\mathbb{R} \times \mathbb{R}^k, 0\big)$$

is non-singular, where $(x, q) = (q_1, \dots, q_k, x_1, \dots, x_n) \in (\mathbb{R}^k \times \mathbb{R}^n, 0)$. In this case we have a smooth $(n-1)$-dimensional submanifold,

$$\Sigma_*(F) = \big\{(q, x) \in \big(\mathbb{R}^k \times \mathbb{R}^n, 0\big) \;\big|\; F(q, x) = F_{q_1}(q, x) = \cdots = F_{q_k}(q, x) = 0\big\}$$

and a map germ $L_F : (\Sigma_*(F), 0) \to PT^*\mathbb{R}^n$ defined by $L_F(q, x) = (x, [F_{x_1}(q, x) : \cdots : F_{x_n}(q, x)])$ is a Legendrian immersion germ. Then we have the following fundamental theorem of Arnol'd and Zakalyukin [1]:

Theorem 5.1. *All Legendrian submanifold germs in $PT^*\mathbb{R}^n$ are constructed by the above method.*

We call F a *generating family* of $L_F(\Sigma_*(F))$. We apply this method to the height function $H : U \times S^n \to \mathbb{R}$ on hypersurface $\boldsymbol{x} : U \to S_n$. By Proposition 4.1, we have

$$\Sigma_*(H) = \big\{(u, V) \in U \times S^n \;\big|\; V = \pm\boldsymbol{e}(u)\big\}.$$

Moreover, we can show that

$$L_H\big(u, \boldsymbol{e}(u)\big) = \big(\boldsymbol{e}(u), \big[x_0(u)e_1(u) - x_1(u)e_0(u) : \cdots : x_0(u)e_{n-1}(u) - x_{n-1}(u)e_0(u)\big]\big).$$

We can also show that the following proposition:

Proposition 5.2. *The height function $H : U \times S^n \to \mathbb{R}$ is a Morse family of hypersurfaces at any point (u, V).*

Proof. For any $V = (v_0, v_1, \dots, v_n) \in S^n$, we have $v_0^2 + v_1^2 + \cdots + v_n^2 = 1$. We consider an open subset $U_0^+ = \{V \in S^n \mid v_0 > 0\} \subset S^n$. Then we have

$$v_0 = \sqrt{1 - v_1^2 - \cdots - v_n^2}$$

on U_0^+ and the height function H is represented by

$$H(u, V) = x_0(u)\sqrt{1 - v_1^2 - \cdots - v_{n+1}^2} + x_1(u)v_1 + \cdots + x_{n+1}(u)v_{n+1}$$

where $\boldsymbol{x}(u) = (x_0(u), x_1(u), \dots, x_n(u))$. We have to prove that the mapping

$$\Delta^* H = (H, H_{u_1}, \dots, H_{u_{n-1}})$$

is non-singular at any point of $\Sigma_*(H)$. We adopt the coordinate neighborhood $(U_0^+, (v_1, \dots, v_n))$ of S^n. By definition, we have

$$\begin{aligned}
&\frac{\partial H}{\partial u_i}(u, V) = \boldsymbol{x}_{u_i}(u) \cdot V, \qquad \frac{\partial H}{\partial v_i}(u) = -x_0(u)\frac{v_i}{v_0} + x_i(u), \\
&\frac{\partial^2 H}{\partial u_i \partial u_j}(u, V) = \boldsymbol{x}_{u_i u_j}(u) \cdot V, \qquad \frac{\partial^2 H}{\partial v_i \partial u_j}(u) = -x_{0 u_j}(u)\frac{v_i}{v_0} + x_{i u_j}(u).
\end{aligned}$$

Therefore the Jacobian matrix of $\Delta^* H$ is given as follows:

$$\begin{pmatrix}
\boldsymbol{x}_{u_1} \cdot V & \cdots & \boldsymbol{x}_{u_{n-1}} \cdot V & -x_0 \frac{v_1}{v_0} + x_1 & \cdots & -x_0 \frac{v_n}{v_0} + x_n \\
\boldsymbol{x}_{u_1 u_1} \cdot V & \cdots & \boldsymbol{x}_{u_1 u_{n-1}} \cdot V & -x_{0u_1} \frac{v_1}{v_0} + x_{1u_1} & \cdots & -x_{0u_1} \frac{v_n}{v_0} + x_{nu_1} \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
\boldsymbol{x}_{u_{n-1} u_1} \cdot V & \cdots & \boldsymbol{x}_{u_{n-1} u_{n-1}} \cdot V & -x_{0u_{n-1}} \frac{v_1}{v_0} + x_{1u_{n-1}} & \cdots & -x_{0u_{n-1}} \frac{v_n}{v_0} + x_{nu_{n-1}}
\end{pmatrix}.$$

We now show that the determinant of the matrix

$$A = \begin{pmatrix}
-x_0 \frac{v_1}{v_0} + x_1 & \cdots & -x_0 \frac{v_n}{v_0} + x_n \\
-x_{0u_1} \frac{v_1}{v_0} + x_{1u_1} & \cdots & -x_{0u_1} \frac{v_n}{v_0} + x_{nu_1} \\
\vdots & \vdots & \vdots \\
-x_{0u_{n-1}} \frac{v_1}{v_0} + x_{1u_{n-1}} & \cdots & -x_{0u_{n-1}} \frac{v_n}{v_0} + x_{nu_{n-1}}
\end{pmatrix}$$

dose not vanish at $(u, V) \in \Sigma_*(H)$. In this case, $V = \pm\boldsymbol{e}$ and we denote

$$a = \begin{pmatrix} x_0 \\ x_{0u_1} \\ \vdots \\ x_{0u_{n-1}} \end{pmatrix}, \qquad b_1 = \begin{pmatrix} x_1 \\ x_{1u_1} \\ \vdots \\ x_{1u_{n-1}} \end{pmatrix}, \quad \dots, \quad b_n = \begin{pmatrix} x_n \\ x_{nu_1} \\ \vdots \\ x_{nu_{n-1}} \end{pmatrix}.$$

Then we have

$$\begin{aligned}
\det A &= \det\left(-a\frac{v_1}{v_0} + b_1, \dots, a\frac{v_n}{v_0} + b_n\right) \\
&= \det\left(-a\frac{v_1}{v_0}, \dots, a\frac{v_n}{v_0} + b_n\right) + \det\left(b_1, \dots, a\frac{v_n}{v_0} + b_n\right) \\
&\ \vdots \\
&= \frac{v_0}{v_0}\det(b_1\ \dots\ b_n) - \frac{v_1}{v_0}\det(ab_2 \dots b_n) - \cdots - \frac{v_n}{v_0}\det(b_1 \dots b_{n-1}a).
\end{aligned}$$

On the other hand, by calculation we have

$$\boldsymbol{x} \wedge \boldsymbol{x}_{u_1} \wedge \cdots \wedge \boldsymbol{x}_{u_{n-1}} = \sum_{i=0}^{n} (-1)^i \det(ab_1 \dots \hat{b}_i \dots b_n).$$

Therefore we have

$$\begin{aligned}
\det A &= \left(\frac{v_0}{v_0}, \ldots, \frac{v_n}{v_0}\right) \cdot (\boldsymbol{x} \wedge \boldsymbol{x}_{u_1} \wedge \cdots \wedge \boldsymbol{x}_{u_{n-1}}) \\
&= \frac{1}{v_0}\boldsymbol{e} \cdot \boldsymbol{e}\|\boldsymbol{x} \wedge \boldsymbol{x}_{u_1} \wedge \cdots \wedge \boldsymbol{x}_{u_{n-1}}\| \\
&= \frac{1}{v_0}\|\boldsymbol{x} \wedge \boldsymbol{x}_{u_1} \wedge \cdots \wedge \boldsymbol{x}_{u_{n-1}}\| \\
&\neq 0.
\end{aligned}$$

For the other local coordinates, we have the similar result to the above calculation. Therefore $\Delta^* H = (H, H_{u_1}, \ldots, H_{u_{n-1}})$ is non-singular on $\Sigma_*(H)$. $\square$

We now show that H is a generating family of $\mathcal{L} \in \Delta$.

Theorem 5.3. *For any hypersurface $\boldsymbol{x} : U \to S_n$, the height function $H : U \times S^n \to \mathbb{R}$ of $\boldsymbol{x}$ is a generating family of the Legendrian immersion $\mathcal{L}$.*

Proof. Let $\pi : PT^*S^n \to S^n$ is a projective cotangent bundle of S^n. The canonical contact form of PT^*S^n is given as follows: We consider the same local coordinate $(U_0^+, (v_1, \ldots, v_n))$ of S^n as in the proof of Proposition 5.3. The corresponding homogeneous coordinate of PT^*S^n by $(v_1, \ldots, v_n, [\zeta_1 : \cdots : \zeta_n])$. Then the canonical contact form of PT^*S^n is given by

$$\alpha = d\boldsymbol{v}_i + \sum_{j \neq i} \frac{\zeta_j}{\zeta_i} d\boldsymbol{v}_j$$

with the affine coordinate on $\{\zeta_i \neq 0\}$. Here we define a map Φ from $\Delta|(U_0^+ \times S^n)$ to the projective cotangent bundle on PT^*S^n by

$$\Phi(\boldsymbol{v}, \boldsymbol{w}) = \big(\boldsymbol{w}, [v_0 w_1 - v_1 w_0 : \cdots : v_0 w_n - v_n w_0]\big).$$

The pull-back of the canonical contact form by Φ is

$$\begin{aligned}
\Phi^*\alpha &= dw_1 + \sum_{i=2}^{n} \frac{v_0 w_i - v_i w_0}{v_0 w_1 - v_1 w_0} dw_i, \\
(v_0 w_1 - v_1 w_0)\Phi^*\alpha &= \sum_{i=1}^{n} (v_0 w_i - v_i w_0) dw_i \\
&= \sum_{i=1}^{n} v_i w_0 dw_i + \sum_{i=1}^{n} v_0 w_i dw_i \\
&= w_0 \left(\sum_{i=1}^{n} v_i dw_i + v_0 \sum_{i=1}^{n} \frac{w_i}{w_0} dw_i \right) \\
&= w_0 \left(\sum_{i=1}^{n} v_i dw_i + v_0 dw_0 \right) \\
&= w_0 \theta_1.
\end{aligned}$$

Thus, Φ is a contact morphism.

Since the height function $H : U \times S^n \to \mathbb{R}$ is a Morse family of hypersurfaces, we have a Legendrian immersion $L_H : \Sigma_*(H) \to PT^*S^n$ defined by

$$L_H\big(u, \boldsymbol{e}(u)\big) = \big(\boldsymbol{e}(u), \big[x_0(u)e_1(u) - x_1(u)e_0(u) : \cdots : x_0(u)e_{n-1}(u) - x_{n-1}(u)e_0(u)\big]\big).$$

Therefore we have $\Phi \circ \mathcal{L}(u) = L_H(u, \boldsymbol{e}(u))$. This means that H is a generating family of $\mathcal{L}(U) \in \Delta$ through Φ. $\square$

We call $\mathcal{L}$ the *Legendrian lift* of the Gauss map $\mathbb{G}$.

In the end of this section, we review the following result on the theory of Legendrian singularities. By the uniqueness result of the $\mathcal{K}$-versal deformation of a function germ, Proposition A.2 and Theorem A.3 in Izumiya [3], we have the following classification result of Legendrian stable germs.

Proposition 5.4. *Let $F, G : (\mathbb{R}^k \times \mathbb{R}^n, 0) \to (\mathbb{R}, 0)$ be Morse families. Suppose that L_F and L_G are Legendrian stable. Then the following conditions are equivalent*:

1. $(W(L_F), 0)$ *and* $(W(L_G, 0))$ *are diffeomorphic as set germs*;
2. L_F *and* L_G *are Legendrian equivalent*;
3. F *and* G *are* P–$\mathcal{K}$*-equivalent*;
4. $f = F|\mathbb{R}^k \times \{0\}$ *and* $g = G|\mathbb{R}^k \times \{0\}$ *are* $\mathcal{K}$*-equivalent.*

6. Contact with great hyperspheres and generic property in low-dimension

In this section we consider the geometric meaning of singularities of the Gauss map $\mathbb{G}$ from the contact view point, and consider generic properties of hypersurface in S^n and show that the assumption of the theorem of this section is generic in the case of when $6 \leqslant n$. Montaldi [5] interpreted the contact of submanifolds in terms of the singularity theory of map germs. Here we quickly review the theory and apply to our case. Let X_i and Y_i $(i = 1, 2)$, be submanifolds of $\mathbb{R}^n$ with $\dim X_1 = \dim X_2$, $\dim Y_1 = \dim Y_2$. We say that the *contact* of X_1 and Y_1 at y_1 is the *same type* as the *contact* of X_2 and Y_2 at y_2 if there is a diffeomorphic germ $\Phi : (\mathbb{R}^n, y_1) \to (\mathbb{R}^n, y_2)$ such that $\Phi(X_1, y_1) = (X_2, y_2)$ and $\Phi(Y_1, y_1) = (Y_2, y_2)$. In this case we write $K(X_1, Y_1; y_1) = K(X_2, Y_2; y_2)$. In the definition, $\mathbb{R}^n$ can be replaced to any manifold. In his paper [5] Montaldi gives a characterization of the notion of contact by using a notion of the singularity theory.

Theorem 6.1 *(Montaldi). Let X_i and Y_i $(i = 1, 2)$ be submanifolds of $\mathbb{R}^n$ with* $\dim X_1 = \dim X_2$, $\dim Y_1 = \dim Y_2$. *Let $f_i : (\mathbb{R}^n, y_i) \to (\mathbb{R}^p, 0)$ be submersion germs with $(f_i^{-1}(0), y_i) = (Y_i, y_i)$ and $g_i : (X_i, x_i) \to (\mathbb{R}^n, y_i)$ immersion germs. Then $K(X_1, Y_1; y_1) = K(X_2, Y_2; y_2)$ if and only if $f_1 \circ g_1$ and $f_2 \circ g_2$ are $\mathcal{K}$-equivalent. For the definition of the $\mathcal{K}$-equivalence, see Izumiya* [3].

For a given vector $V \in S^n$, we define a function $\mathfrak{h}_V : S^n \to \mathbb{R}$ by $\mathfrak{h}_V(W) = W \cdot V$. By definition, we have $\mathfrak{h}_V^{-1}(c) = S^{n-1}(V, c)$. For any $u_0 \in U$ and $V_0 = \pm\mathbb{G}(u_0)$, we have

$$\mathfrak{h}_{V_0} \circ \boldsymbol{x}(u_0) = H\big(u_0, \pm\mathbb{G}(u_0)\big) = 0,$$
$$\frac{\partial \mathfrak{h}_{V_0} \circ \boldsymbol{x}}{\partial u_i}(u_0) = \frac{\partial H}{\partial u_i}\big(u_0, \pm\mathbb{G}(u_0)\big) = 0 \quad (i = 1, \ldots, n).$$

This means that the great hypersphere $\mathfrak{h}_{V_0}^{-1}(0) = S^{n-1}(V_0, 0)$ is tangent to the hypersurface $\boldsymbol{x} : U \to S\ n$ at $p = \boldsymbol{x}(u_0)$. In this case we call $S^{n-1}(V_0, 0)$ *tangent great hypersphere* of $M = \boldsymbol{x}(U)$ at $p = \boldsymbol{x}(u_0)$. In Euclidean space, if images of the Gauss map of a hypersurface at two point $u_1, u_2 \in U$ are the same, then tangent hyperplanes at these points are parallel. However the situation in S^n is different from the Euclidean case as the following lemma shows:

Lemma 6.2. *Let $\boldsymbol{x} : U \to S^n$ be a hypersurface and u_1, u_2 are two points in U. Then $\mathbb{G}(u_1) = \pm\mathbb{G}(u_2)$ if and only if $S^{n-1}(V_1, 0) = S^{n-1}(V_2, 0)$, where $V_i = \mathbb{G}(u_i)$.*

We now have tools for the study of the contact between hypersurfaces and great hyperspheres. Let $\mathbb{G}_i : (U, u_i) \to (S^n, \boldsymbol{e}_i(u_i))$ $(i = 1, 2)$ be Gauss map germs of hypersurface germs $\boldsymbol{x}_i : (U, u_i) \to (S^n, \boldsymbol{x}_i(u_i))$ and $\mathcal{L}_i = (\boldsymbol{x}_i, \boldsymbol{e}_i) : (U, u_i) \to \Delta$ be corresponding Legendrian immersion germs. By Zakalyukin [8], if the regular set of $\mathbb{G}_i = \pi_2 \circ \mathcal{L}_i$ is dense in (U, u_i) for both $i = 1, 2$, then $(\mathbb{G}_1(U), \boldsymbol{e}(u_1))$ and $(\mathbb{G}_2(U), \boldsymbol{e}(u_2))$ are diffeomorphic as set germs if and only if $\mathcal{L}_1$ and $\mathcal{L}_2$ are Legendrian equivalent. Arnol'd [1] and Zakalyukin [7], this condition is also equivalent to the condition that two generating families H_1 and H_2 are P–$\mathcal{K}$-equivalent. Here, $H_i : (U \times S^n, (u_i, \boldsymbol{e}(u_i))) \to \mathbb{R}$ is the height function germ of $\boldsymbol{x}_i$.

On the other hand, we denote $h_i(u) = H_i(u, \boldsymbol{e}(u_i))$, then we have $h_i(u) = \mathfrak{h}_{\boldsymbol{e}(u_i)} \circ \boldsymbol{x}_i(u)$. By Theorem 6.1,

$$K\big(x_1(U), S^{n-1}\big(\boldsymbol{e}(u_1), 0\big); x_1(u_1)\big) = K\big(x_2(U), S^{n-1}\big(\boldsymbol{e}(u_2), 0\big); x_2(u_2)\big)$$

if and only if h_1 and h_2 are $\mathcal{K}$-equivalent. Therefore, we can apply the arguments in the appendix to our situation.

Theorem 6.3. *Let $\boldsymbol{x}_i : (U, u_i) \to (S^n, \boldsymbol{x}_i(u_i))$ $(i = 1, 2)$ be hypersurface germs such that the corresponding Legendrian immersion germs $\mathcal{L}_i = (\boldsymbol{x}_i, \boldsymbol{e}_i) : (U, u_i) \to (\Delta, (\boldsymbol{x}_i(u_i), \boldsymbol{e}_i(u_i)))$ are Legendrian stable. Then the following conditions are equivalent*:

1. $(\mathbb{G}_1(U), \boldsymbol{e}(u_1))$ *and* $(\mathbb{G}_2(U), \boldsymbol{e}(u_2))$ *are diffeomorphic as set germs*;
2. $\mathcal{L}_1$ *and* $\mathcal{L}_2$ *are Legendrian equivalent*;
3. H_1 *and* H_2 *are* P–$\mathcal{K}$*-equivalent*;
4. h_1 *and* h_2 *are* $\mathcal{K}$*-equivalent*;
5. $K(x_1(U), S^{n-1}(\boldsymbol{e}(u_1), 0); x_1(u_1)) = K(x_2(U), S^{n-1}(\boldsymbol{e}(u_2), 0); x_2(u_2))$.

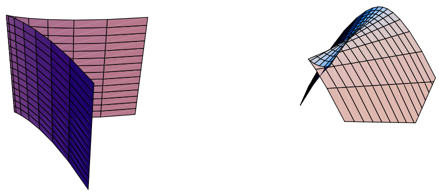

Fig. 1.

Proof. We remark that $(\mathbb{G}_i(U), \boldsymbol{e}(u_i)) = (W(\mathcal{L}_i), \boldsymbol{e}(u_i))$. Since both of $\mathcal{L}_i$ are Legendrian stable, these satisfy the assumption of Proposition 5.4, then the condition 1, the condition 2 and the condition 3 are equivalent. It also follows from the theory of Arnol'd and Zakalyukin [1,7] that H_i is $\mathcal{K}$-versal deformation of h_i. By the uniqueness result of the $\mathcal{K}$-versal deformation of a function germ, the conditions 3 and 4 are equivalent. Finally, by the previous argument (mainly from Theorem 6.1), we have the equivalence between the conditions 4 and 5. $\square$

We remark that the above theorem gives an interpretation of the geometric meaning of the image of the Gauss map.

Moreover we can show that the assumption of Theorem 6.3 is generic in the case of when $6 \leqslant n$ by using a transversality theorem by Wassermann [6] (the arguments are the same as those of Section 6 in Izumiya, Pei and Takahashi [2], so that we omit it).

Proposition 6.4. *Suppose that $n \leqslant 6$. Then there exists an open dense subset $\mathcal{O} \subset Emb(U, S^n)$ such that for any $\boldsymbol{x} \in \mathcal{O}$, the germ of the Legendrian lift $\mathcal{L}$ of the Gauss map at each point is Legendrian stable.*

7. Surfaces in 3-sphere

In the case when $n = 3$, Miyawaki [4] investigated a surface $\boldsymbol{x} : U \to S^3$ as an application of the singularity theory. However it is written in Japanese, so that we briefly review his results in this section. In this case we call $S^2(V, 0)$ a *great sphere*. By the classification of function germs (cf., [1]), we have the following theorem:

Proposition 7.1. *([4, Proposition 4.3]) Let $\boldsymbol{x} : U \to S^3$ be a surface. If the germ of the height function on the surface at a point $(u_0, V_0) \in U \times S^3$ is $\mathcal{K}$-versal deformation of $h_{V_0}(u) = H(u, V_0)$, then the image of the Gauss map germ $\mathbb{G} : (U, u_0) \to S^3$ is diffeomorphic to one of the following germs: the plane, the cuspidal edge or the swallowtail.*

Here, the *cuspidal edge* is a set germ parametrized by (u_1, u_2^2, u_2^3) and the *swallowtail* is parametrized by $(3u_1^4 + u_1^2u_2, 4u_1^3 + 2u_1u_2, u_2)$ (cf., Fig. 1).

Since $n = 3 \leqslant 6$, the assumption of the above proposition is generic by Proposition 7.2.

Theorem 7.2. *([4, Theorem 5.4]) Let $\boldsymbol{x} : U \to S^3$ be a surface. We assume that the germ of height function at the point (u_0, V_0) is $\mathcal{K}$-versal deformation of h_{V_0} and the image of the Gauss map is diffeomorphic to the cuspidal edge. Then we have the following:*

1. *The parabolic set $K^{-1}(0)$ is a regular curve. Moreover, if $K = \kappa_1\kappa_2 = 0$ and $\kappa_1 = 0$, then the principal direction corresponding to $\kappa_1 = 0$ is transverse to $K^{-1}(0)$.*
2. *The intersection of the surface $\boldsymbol{x}$ and tangent great sphere at $\boldsymbol{x}(u_0, V_0)$ is locally diffeomorphic to the ordinary cusp $\{(u_1, u_2) \mid u_1^3 \pm u_2^2 = 0\}$.*

Theorem 7.3. *([4, Theorem 5.5]) Let $\boldsymbol{x} : U \to S^3$ be a surface. We assume that the germ of height function at the point (u_0, V_0) is $\mathcal{K}$-versal deformation of h_{V_0} and the image of the Gauss map is diffeomorphic to the swallowtail. Then we have the following:*

1. *The parabolic set $K^{-1}(0)$ is a regular curve. Moreover, if $K = \kappa_1\kappa_2 = 0$ and $\kappa_1 = 0$, then the principal direction corresponding to $\kappa_1 = 0$ is transverse to $K^{-1}(0)$ expect at (u_0, V_0) and it is tangent to $K^{-1}(0)$ at (u_0, V_0).*
2. *The intersection of the surface $\boldsymbol{x}$ and tangent great sphere at $\boldsymbol{x}(u_0, V_0)$ is locally diffeomorphic to the tachnodal $\{(u_1, u_2) \mid u_1^4 - u_2^2 = 0\}$.*
3. *For any $\epsilon \in \mathbb{R}$, there exist two different points $u, u' \in U$ such that $|u_0 - u| < \epsilon$, $|u_0 - u'| < \epsilon$, neither of u nor u' is a parabolic point, and the tangent great sphere of $M = \boldsymbol{x}(U)$ at u and u' are the same.*

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